

## Comment on MOTION OF AN ELECTRON IN THE FIELD OF A BINOMIAL POTENTIAL OF A PROTON

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It has been shown that the binomial potential

$$V(r) = -\frac{e^2}{r} + \frac{\Gamma}{r^2}, \quad \text{with} \quad \Gamma = \frac{\hbar^2}{2m_e}$$

for proton-electron interaction cannot be accepted from physical reasons and that it leads to predictions that are in definite contradiction with experiment of atomic physics.

### 1 Preliminary remarks

V.K. Gudym and E.V. Andreeva, in their paper *Motion of an electron in the field of a binomial potential of a proton*, assumed a binomial potential of the form

$$V(r) = -\frac{e^2}{r} + \frac{\Gamma}{r^\gamma}, \quad \gamma = 2, \quad \Gamma = \frac{\hbar^2}{2m}$$

to describe electron-proton interaction. They discussed in the framework of Newtonian mechanics its effect on shapes of electron's bounded

classical orbits as well as on scattering of high energy electrons (with energies  $40 - 400MeV$ ). The particular values of  $\gamma$  and  $\Gamma$  were specified by the requirement that the potential should have its minimum numerically equal to energy of the ground state of Bohr's atom  $E_B \approx -13.6eV$  and be located at the Bohr's radius  $r_B \approx 5.3 \times 10^{-9}cm$ , where, as usual,

$$E_B = -\frac{1}{2}\alpha^2 m_e c^2, \quad r_B = \alpha^{-1} \frac{\hbar}{m_e c}, \quad \alpha = \frac{e^2}{\hbar c}.$$

Their requirement had originated, as they wrote, from *'the indisputable fact that an atomic electron is constantly located at some distance from the nucleus'*. Unfortunately, the Authors did not explain in more detail the mysterious statement. Therefore, to clarify that point, note that the shape of the binomial potential's well suggest one should expect the electron's ground state wave function to be spherically symmetric with respect to the centre and concentrated mainly in the vicinity of a spherical shell of radius  $r_B$ . Recall that electron's kinetic energy in the ground state must be bounded from zero, thus quantum-mechanically the ground state electron is not at rest at all, in contrast to what the Authors expected. We shall come to the issue in the next section in which we solve Schrödinger equation with the binomial potential.

In their paper the Authors focused mainly on examining the classical orbits of electrons in the binomial potential in the framework of Newtonian mechanics. Calculation of classical orbits of trial bodies in such potential (with arbitrary constants in place of  $\Gamma$  and  $e$ ) is a very old (celestial mechanics) and now standard graduate textbook problem, thus the Author's calculations introduce nothing new, the more that, as it will be shown, the particular binomial potential would lead to effects that would be not observable in atomic physics. Moreover, the Author's using of Newtonian mechanics for scattering of ultra-relativistic electrons with energy  $40 - 400MeV$ , thus  $80 - 800$  times greater than the electron's rest mass, cannot be accepted. For such electrons the relativistic dynamics must be necessarily used.

Another weak point is that the Authors have not explained the nature of the  $\Gamma$  term. With no such information given, one can only surmise from the fact of using atomic quantities in specifying the term, that it should have electromagnetic origin, yet it would be an

exaggeration to expect the term could be caused by gravitational or even short range forces such as weak interactions, in the latter case the potential would have to be damped exponentially, however. In what follows, we shall therefore be assuming that the  $\Gamma$  term has electromagnetic origin.

The next deficiency of the paper is a lack of physical justification for such a fundamental modification of the proton-electron interaction. Various kinds of corrections to the Coulomb field are interpreted as effective potentials arising in the presence of other charges in atomic interiors or in ambient space of, say, crystals. Effective potentials of this kind have no fundamental nature and vanish for isolated proton-electron pairs. The proposed  $\Gamma$  term is spherically symmetric and has a dipole-like falloff. Such term, however, cannot originate from proton's internal structure because dipolar potential is angle dependent – in spherical coordinates it is proportional to  $r^{-2} \cos \theta$  – and due to a complicated internal proton's dynamics its multipolar structure is rather a zero mean value fluctuation field. Higher multipolar momenta falloff quicker so they cannot produce the  $\Gamma$  term, as well. It cannot be excluded a possibility that a  $\Gamma$ -like term, possibly more complicated than merely  $r^{-2}$ , still could be considered as an attempt to a naive and very approximate description of an effective correction to the Coulomb field at larger radii due to, say, internal electromagnetic structure of the proton. However, such a correction should have only a very tiny effect on energy structure of the hydrogen, and could be derived only in the framework of quantum field theory.

The other weak point is the Authors have not specified the region of validity of the  $\Gamma$  term (e.g. by giving a cutoff radius). Their calculations suggest they assumed validity of the binomial potential globally, moreover, they claimed the term was suitable also for modelling internal atomic structure. However, the term cannot be correct in whole space. Note that the charge density  $-(4\pi)^{-1} \vec{\nabla}^2 U$  corresponding to the  $\Gamma$  term with  $U = -(\Gamma/e)r^{-2}$  does not make sense as a global charge distribution. Electric charge outside a sphere of radius  $a$  equals  $\Gamma/(ae)$ , while total charge enclosed by the sphere is the opposite. Both charges sum up to zero, and in the limit  $a \rightarrow 0$  are infinite (so in the centre there is concentrated infinite negative charge). Another way of seeing this (this shows also distributional

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character of the charge density corresponding to the  $\Gamma$  term) can be inferred by inspection of a regularized profile  $-(\Gamma/e)(r^2 + \varepsilon^2)^{-1}$  of the  $\Gamma$  term, which leads to a smooth charge density

$$\frac{\Gamma (r^2 - 3\varepsilon^2)}{2e\pi (r^2 + \varepsilon^2)^3}$$

negative for  $0 < r < \sqrt{3}\varepsilon$  and positive elsewhere. Contribution to total charge from each region is  $\mp 3\sqrt{3}\Gamma(8e\varepsilon)^{-1}$ , respectively, and infinite in the limit  $\varepsilon \rightarrow 0$ . This shows the  $\Gamma$  term should be somehow modified for small radii, note however, that outside a sphere of radius  $a$  there would still reside  $er_B/a$  of positive charge (where it would come from then?).

A serious omission is that the Authors have not assessed the effect of the  $\Gamma$  term on energy of photons emitted by hydrogen, which is the question that naturally arises and that allows one to decide whether or not such correction to the Coulomb field is physically viable in atomic interiors. By construction, the Coulomb and  $\Gamma$  terms of the binomial potential are comparable in the vicinity of Bohr's radius. One should therefore expect significant reorganization of energy structure of hydrogen as wave functions must penetrate also regions where the  $\Gamma$  term is dominant and strongly repulsive. That such assessment requires quantum-mechanical arguments is best seen from the following observation: in classical picture the lowest energy electron in the field of the binomial potential, which has the global minimum, is at rest at Bohr's radius and has energy  $-13.6\text{ eV}$ , thus one would expect  $13.6\text{ eV}$  for ionization energy, the same as for ordinary hydrogen, and this is what the Authors expected. Quantum mechanically, however, it suffices only  $5.13\text{ eV}$  for ionization! Accidentally, the semiclassical Bohr's model with pure Coulomb field yielded the same energy spectrum as the one predicted quantum mechanically! For the binomial potential such correspondence of the two pictures is no longer valid. As an aside, we remark that the Bohr's quantization condition  $m_e v_r r = n\hbar$  for circular orbits in the binomial potential gives

$$E_n = \frac{E_B}{n^2 + \frac{2m\Gamma}{\hbar^2}}$$

for energy levels, and  $n = 0$  is also allowable if  $\Gamma \neq 0$ . This simplified

calculation shows that if  $\Gamma$  was much less than  $\hbar^2/(2m_e)$  one would expect energy levels to be only slightly different from those of ordinary hydrogen for  $n \neq 0$ , while one should expect large departures from the energy spectrum for  $\Gamma = \hbar^2/(2m_e)$ . As we shall see in the next section the correct non-relativistic quantum mechanical calculation yields much more complicated energy spectrum than the one above, and also that the modification of proton-electron interaction by the  $\Gamma$  term, the Authors have proposed, leads to predictions that are in severe contradiction with experiment.

## 2 Eigenvalue problem for the binomial potential

In what follows we shall find bound states of electron in the binomial potential and the corresponding energy spectrum. We shall refer to the hydrogen atom with the modified Coulomb potential as 'Γ-hydrogen'.

The stationary Schrödinger equation  $\hat{H}\Psi = E\Psi$  with the binomial potential reduces to

$$\left(-\partial_x^2 - \frac{2}{x}\partial_x + \frac{1}{x^2}\hat{L}^2 - \frac{2}{x} + \frac{1}{x^2} + \epsilon\right)\Psi = 0,$$

with

$$\hat{L}^2 = -\left(\partial_\theta^2 + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\right),$$

where we have defined dimensionless radius  $x$  and dimensionless energy  $\epsilon$

$$r = r_B \cdot x, \quad E = E_B \cdot \epsilon.$$

As the differential operators  $\hat{H}$ ,  $\hat{L}^2$  and  $\hat{L}_z = -i\partial_\phi$  commute with each other, the general solution reads

$$\Psi = R(x)Y_{lm}(\theta, \phi), \quad l = 0, 1, 2, \dots, \quad m = 0, \pm 1, \dots \pm l,$$

where  $Y_{lm}(\theta, \phi)$  are spherical harmonics and the radial function  $R(x)$  is a corresponding solution of equation

$$R_{,xx} + \frac{2}{x}R_{,x} - \frac{1+l(l+1)}{x^2}R + \frac{2}{x}R - \epsilon R = 0.$$

For bound states we assume  $\epsilon > 0$  (then  $E < 0$ ). By setting  $\beta_l = \frac{1}{2} \left( \sqrt{5 + 4l(l+1)} - 1 \right)$  and  $R(x) = x^{\beta_l} e^{-x\sqrt{\epsilon}} Q(2x\sqrt{\epsilon})$  we obtain the Laguerre equation

$$\xi \ddot{Q}(\xi) + (1 + a - \xi) \dot{Q}(\xi) + n_r Q(\xi) = 0,$$

$$x = \frac{\xi}{2\sqrt{\epsilon}}, \quad a = 1 + 2\beta, \quad n_r = \frac{1}{\sqrt{\epsilon}} - (1 + \beta),$$

of which solutions read  $Q(\xi) = L_{n_r}^{1+2\beta_l}(\xi)$  ( $L$ 's are Laguerre's functions) and are normalizable only if  $n_r = 0, 1, 2, \dots$ . Consequently, we obtain a discrete spectrum of energy levels

$$\epsilon_{l,n_r} = \frac{1}{(1 + \beta_l + n_r)^2}, \quad \beta_l = l + \frac{2}{1 + 2l + \sqrt{5 + 4l(l+1)}},$$

$$n_r = 0, 1, 2, \dots$$

together with the corresponding eigenfunctions

$$\Psi_{(m,l,n_r)}(x, \theta, \phi) =$$

$$= C_{(m,l,n_r)} x^{\beta_l} e^{-x\sqrt{\epsilon_{l,n_r}}} L_{n_r}^{1+2\beta_l}(2x\sqrt{\epsilon_{l,n_r}}) Y_{lm}(\theta, \phi) e^{im\phi}$$

( $C$ 's are normalization constants).

## 2.1 Implications for the binomial potential

As follows from the energy spectrum found in the previous section, the presence of the  $\Gamma$  term removes the characteristic for the pure Coulomb field accidental degeneracy of energy levels of Bohr's atom and, due to the huge value of  $\Gamma = \hbar^2/(2m_e)$ , drastically affects energy structure, such that this change could be easily detected experimentally. In particular, for the ground state, which is spherically symmetric

$$\Psi = C (xe^{-x})^{1/\chi},$$

$$\epsilon = \frac{1}{\chi^2}, \quad C = \frac{1}{4\sqrt{\pi}\Gamma(2\chi)} \left( \frac{2}{\chi} \right)^{\chi+1}, \quad \chi = \frac{1 + \sqrt{5}}{2}$$

( $\chi$  is the famous 'golden ratio'). Consequently, the ionization energy of the  $\Gamma$ -hydrogen is only 5.13 eV while ionization energy of Bohr's

atom is  $13.6 \text{ eV}$ . The ground state of the electron in the  $\Gamma$ -hydrogen, in contrast to what the Authors assumed while determining the binomial potential, is different from the first energy level of the Bohr's model. The reason is that the wave function of the electron in the ground state in the particular binomial potential, penetrates the region of small radii where the repulsive  $\Gamma$  term dominates the attractive Coulomb interaction. Note, that the corresponding probability density has its maximum located on the sphere of one Bohr's radius ( $x = 1$ ), (this in a sense agrees with the Author's intention of having the electron '*constantly located*' at the Bohr's radius, however the electron is not at rest at all as the expectation value of kinetic energy in the ground state is nonzero) and that the probability density is zero in the centre reflecting the dominant repulsive effect of the  $\Gamma$  term (to compare with, the ground state wave function of the ordinary hydrogen is  $e^{-x}/\sqrt{\pi}$  and has its maximum located in the centre).

As concerns the removed degeneracy of energy levels, it predicts, in particular, the transition  $2S$  ( $n_r = 1, l = 0$ )  $\rightarrow$   $2P$  ( $n_r = 0, l = 1$ ) with energy

$$\frac{2\hbar^2}{m_e e^4} (E_{2S} - E_{2P}) = \frac{4}{(1 + \sqrt{13})^2} - \frac{4}{(3 + \sqrt{5})^2} \approx 0.0427$$

(absent for ordinary hydrogen). The eigenfunctions  $2S$  and  $2P$  overlap in the limit  $\Gamma \rightarrow 0$  with eigenfunctions of ordinary hydrogen (the subtle structure of  $2S$  and  $2P$ , that is,  $2^2S_{1/2}$ ,  $2^2P_{1/2}$  and  $2^2P_{3/2}$ , is not seen in non-relativistic description, of course). Photon transitions between  $2S$  and  $2P$  with such large energy are not observed.

Both the above examples emphatically prove that the value of  $\Gamma$  the Authors assumed in the binomial potential is far too large, and the effect of the  $\Gamma$  term could be easily detected if it really existed.

### 3 Summary

It has been shown in the previous sections that the binomial potential

$$V(r) = -\frac{e^2}{r} + \frac{\Gamma}{r^2}, \quad \text{with} \quad \Gamma = \frac{\hbar^2}{2m_e}$$

for proton-electron interaction cannot be accepted from mathematical reasons and that it is definitely excluded by experiment.